

Turbulence Driven by a Deterministic Chaotic Dynamics

D. Volchenkov *

R. Lima[†]

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Abstract

In the inertial range of fully developed turbulence, we model the vertex network dynamics by an iterated unimodular map having the universal behavior. Inertial range anomalous scaling for the pair correlation functions of the velocity and the local energy dissipation is established as a consequence of the chaotic behavior of the unimodular map when the Feigenbaum attractor loses stability. The anomalous corrections determined by the Feigenbaum constant η to the Kolmogorov's spectra are larger than those observed in experiments.

Turbulence Modeling and Turbulence Control in fluid dynamics is a paradigmatic problem and successful techniques in this domain may serve as models to deal with other complex extended systems. It is generally accepted now that a real understanding of turbulence requires calculations and models that are globally sensitive to all length scales.

To elucidate this conviction, let us briefly explain the present situation in the quantum field theory of fully developed turbulence of incompressible fluid which has been developed for the last two decades (see [1] for a review). In order to be specific we consider the correlator $K = \langle v(\mathbf{k})v(-\mathbf{k}) \rangle$ of the velocity field v in the momentum representation ($k \equiv |\mathbf{k}|$ and $\langle \dots \rangle$ denote the time averaging). It is found from the Dyson equation

$$K^{-1} = \nu k^2 + m^2 - \Sigma(k), \quad (1)$$

*Solid State Electronics Department, Research Institute of Physics, St.-Peterburg State University (Russia) and Centre de Physique Theorique, CNRS; e-mail: volchen@cpt.univ-mrs.fr

[†]Centre de Physique Theorique, CNRS, Luminy Case-907, 13288 Marseille (France)

where m is the inverse integral turbulent scale, ν is the viscosity parameter, and $\Sigma(k)$ is an infinite sum of all 1-irreducible diagrams of the Wyld's diagrammatic technique [2]. In the framework of $4-2\varepsilon$ expansion the series (1) takes the form [3]

$$K^{-1} = (\nu k^2 + m^2) \left[1 + \sum_{n=1}^{\infty} (g k^{-2\varepsilon})^n c_n(m/k, \varepsilon) \right], \quad (2)$$

The inverse viscous dissipative length λ plays the role of a maximum momentum in the problem. The Reynolds number $\text{Re} = (\lambda/m)^{4/3}$ is assumed to be large. For $k \sim m \ll \lambda$ and $\varepsilon > 0$, the dimensionless parameter of the expansion $g k^{-2\varepsilon} \sim (k/\lambda)^{-2\varepsilon}/\nu$ in (2) is not small and it is necessary to sum the series. This problem reduces to a determination of the asymptotic value of $K_\chi = K(\chi k, \chi m)$ for $\chi \rightarrow 0$ (everything is fixed except for χ). It is solved by the renormalization group procedure (RG) for any finite ratio m/k [3]-[5].

However, the coefficient c_n in (2) has additional singularities as $m/k \rightarrow 0$ which cannot be handled by RG since they have nothing to do with the scale invariance property (the parameter m does not involve the renormalization). These singularities express the intermittency phenomenon ([6], Chap. 25) that is the actual reason why the bare RG is ineffective in turbulence [7].

The $m/k \rightarrow 0$ asymptotics of the series (2) is found due to the Short Distance Expansion method [8], introduced by Wilson. It leads to the form

$$K(k) = A k^{-11/3} \left[1 + \sum_i b_i (m/k)^{\Delta_{F_i}} \right] \quad (3)$$

where A is the Kolmogorov's constant, b_i is the coefficient analytic in m/k , and Δ_{F_i} is the critical exponent of a set of composite operators F_i . The series in (3) would give the corrections to the leading Kolmogorov's asymptotics if all $\Delta_{F_i} > 0$.

A number of works has been devoted to the computation of Δ_{F_i} for various sets of composite operators in hydrodynamics [5], [9] and magnetohydrodynamics [10]. It is found that there are infinitely many composite operators for which $\Delta_{F_i} \leq 0$. An example of such a "dangerous" one is the composite operator of local energy dissipation $E = -\nu(\partial_i v_k + \partial_k v_i)^2/2$ with $\Delta_E = 0$ [5]. The contributions to (3) due to such operators are not small and it is necessary to

sum the infinite series in (3). There is no reliable technique for such a summation found. To proceed further, remaining of the idea of homogeneously fully developed turbulence, one must assign the statistics to each set of composite operators with $\Delta_{F_i} \leq 0$.

It is conventional now that one has to employ a model, in which all scales are presented equally to achieve the further progress. In this Letter we suggest such a model.

The hydrodynamics of the incompressible fluid is described by the Navier-Stokes equation

$$\nabla_t v_i = \nu \Delta v_i, \quad \nabla_t \equiv \partial_t + (v \partial), \quad (4)$$

where $v_i(\mathbf{x}, t)$ is the transverse ($\partial_i v_i = 0$) velocity field, ν is the viscosity. To meet the energy balance in the system, one has to compensate the viscous dissipation by an energy pump of power W with some spectral density $\mathfrak{d}(k)$,

$$W = \frac{1}{(2\pi)^3} \int d\mathbf{k} \mathfrak{d}(k). \quad (5)$$

In [3]-[5] and in the descendant papers [9]-[10], the pump source has been introduced in (4) by the Gaussian white noise f . The power model for the spectral density $\mathfrak{d}(k) \sim k(k^2 + m^2)^{-\varepsilon}$ chosen in literature defines the Gaussian statistics for f completely, $\langle f f \rangle(k) \sim \mathfrak{d}(k)$. Nevertheless, such a model is not uniquely determined: One can introduce an arbitrary bounded value function $\mathfrak{h}(m/k)$ ($\mathfrak{h}(0) = 1$) into the Gaussian covariance. This function did not involve the renormalization procedure in [3]-[5]. From the physical point of view, this fact means that the field f has some redundant degrees of freedom which do not change to the scaling ones in the process of renormalization.

In actual experiments, the power W pumped into the inertial range of scales $m \ll k \ll \lambda$ is delivered by the large scale vertices entering the system sometime in a variety of sizes ($k \simeq m$), energies, and enstrophies. These vertices dissolve due to the strong nonlinearity of hydrodynamical interactions into ones of the smaller scales ($k > m$) forming the direct energy cascade. Alternatively, a small scale vertex would associate with others in the fractal sets contributing towards the inverse cascade. The resulting picture constitutes a vortex network encoded, at each moment of time, by a *discrete* set of ratios $0 < \{m/k\}_n < 1$. This network has much in common with dynamical systems.

We shall consider the pumping generated by some dynamical system,

$$\mathcal{W}_\tau(t) = \tau^{1/2} \sum_{n=0}^{\lfloor t/\tau \rfloor} z_n. \quad (6)$$

Here $\tau > 0$ is a time step, the square brackets $\lfloor \dots \rfloor$ denote the integer part. z_n are the iterates of a 1-parameter unimodular map [11] $T : \mathbb{X} \rightarrow \mathbb{X}$ of the phase space $\mathbb{X} \equiv [0, 1]$. We also suppose that the map T possesses the invariant measure μ_T .

As an example of such a map, one can consider the triangular map $Tz = r \left(1 - 2 \left| \frac{1}{2} - z \right| \right)$ or the logistic map $Tz = rz(1 - z)$ which is known to describe the angles of a strongly damped kicked rotator. However, the existence of invariant measure for the latter map is not proven for any r except $r = 4$. The dynamics of unimodular maps has been studied extensively last decades. For the values of the control parameter $1 < r < r_\infty$, the relevant Liapunov exponent is always negative (except the bifurcation points r_n when it becomes zero), while for $r_\infty < r$ this exponent is mostly positive indicating chaotic behavior.

The chaotic behavior observed in the unimodular maps results from the successive pitchfork bifurcations, which provides the mechanism for the successive doubling of the fixed points. It is important that the correlations between the j -shifted sequences

$$\mathcal{C}(j) = \lim_{\tau \rightarrow 0} \left[\frac{\tau}{t - t'} \right] \sum_{n=0}^{\lfloor (t-t')/\tau \rfloor} z_n z_{n+j} \quad (7)$$

generated by the unimodular map decay with a power law in j as $r \rightarrow r_\infty$ [12],

$$\mathcal{C}(j) \simeq j^{-\eta} \mathcal{C}(1, \rho) + \mathcal{O}(r - r_\infty). \quad (8)$$

Here η is the universal constant [13], ρ is the “scaling function” invariant with respect to the doubling transformation T^2 .

An external force $f_\tau(\mathbf{x}, t) : \mathbb{R}^{d+1} \times \mathbb{X} \rightarrow \mathbb{R}^{d+1}$, which we introduce in (4) is regarded as a priori arbitrary chaotic process $\dot{\mathcal{W}}_\tau \equiv \tau^{1/2} \sum_{n \geq 0} z_n \delta(t - n\tau)$, $z_n \in \mathbb{X}$, generated by a *deterministic* chaotic dynamics of the map T (d is the space dimensionality).

In the language of quantum field theory, the time step $\tau > 0$ plays the role of the limiting time scale, dividing the statistics of “fast” ($t < \tau$) and “slow” ($t > \tau$) modes. It is evident that the statistics

of the fast modes $\mathbf{f}(\mathbf{x}, t < \tau)$ is not affected by the dynamics of the map T , so that one can treat as a Gaussian white noise with the covariance

$$D_{rs}(\mathbf{x} - \mathbf{y}, t - t') = (2\pi)^{-3} \int d\omega \int d\mathbf{k} P_{rs} \mathfrak{d}(k) e^{i[\mathbf{k}(\mathbf{x}-\mathbf{y}) - \omega(t-t')]} \quad (9)$$

where $P_{sr} = \delta_{sr} - k_s k_r / k^2$ is the transversal projector. Following [1], we chose the spectral density in the power form $\mathfrak{d}(k) = g\nu^3 k^{4-d-2\varepsilon}$, in which g is a coupling constant. The actual value of the parameter of regular expansion is $\varepsilon = 2$, so that $\mathfrak{d}(k)$ represents a power model for $\delta(\mathbf{k})$ located in the infrared ($k \simeq 0$) region. The other way round, the statistics of the slow modes $\mathbf{f}_\tau(\mathbf{x}, t > \tau)$ is essentially non-Gaussian.

Then the velocity field correlator $K = \langle \mathbf{v}(\mathbf{x}, t) \mathbf{v}(\mathbf{y}, t') \rangle$ factors in

$$K = \mathfrak{K}(\mathbf{x} - \mathbf{y}, t - t') \sum_{n \geq 2} \mathcal{C}_{\tau n}(t - t'; \mathbf{q}_1, \dots, \mathbf{q}_n), \quad t > t'. \quad (10)$$

Here $\mathfrak{K}(\mathbf{x} - \mathbf{y}, t - t')$ is the correlator in the fast modes Gaussian statistics, and the infinite sum describes the chaotic process as $t > \tau$. All correlations

$$\mathcal{C}_{\tau n} = \lim_{\tau \rightarrow 0} \left[\frac{\tau}{t - t'} \right] \sum_{\mathbf{p} > 0}^{[t-t'/\tau]} z_{\mathbf{p}} z_{\mathbf{p}+\mathbf{q}_1} \dots z_{\mathbf{p}+\mathbf{q}_n}, \quad \mathbf{q}_k \equiv \left[\frac{t - t_k}{\tau} \right] \in \mathbb{Z}, \quad (11)$$

($t > \dots > t_k > t_{k-1} > \dots > t' > 0$) exist since there is the continuous invariant measure μ_T .

The correlator (10) is finite for any ratio m/k fixed: \mathfrak{K} is found from the Dyson equation (1) in the framework of some quantum field theory [1] with the multiplicatively renormalized action functional [5]

$$S_{\mathbf{R}} = \frac{1}{2} \mathfrak{M}^{2\varepsilon} \int \int d\mathbf{k} dt \mathbf{v}' D \mathbf{v}' + \int \int d\mathbf{k} dt \mathbf{v}' [-\partial_t \mathbf{v} - (v\partial) \mathbf{v} + \nu Z_\nu \Delta \mathbf{v}] \quad (12)$$

of the basic \mathbf{v} and auxiliary \mathbf{v}' fields. D is the covariance (9). The only renormalization constant $\nu_0 = \nu Z_\nu$ is required to subtract singularities, \mathfrak{M} is the renormalization mass parameter. All correlation functions of the renormalized quantum field theory are finite for any

fixed m/k . In the large-scale asymptotics $s \equiv k/\mathfrak{M} \rightarrow 0$, in $d = 3$, they demonstrate the scaling behavior with the Kolmogorov's critical indices of velocity $\Delta_v = -1/3$ and time $\Delta_t = -2/3$. As we have mentioned above, in the limit $m/k \rightarrow 0$, the additional singularities arise in \mathfrak{K} . We demonstrate, however, that (10) is still finite.

In the majority of experimental data, the correlator K and the correlation function of the local energy dissipation $E \equiv \langle \Phi(\mathbf{x})\Phi(\mathbf{y}) \rangle$, $\Phi(\mathbf{x}) = \partial_i v_j(\mathbf{x})\partial_j v_i(\mathbf{x})$, have a power law behavior

$$K(k) \sim k^{-(11/3+\delta)}, \quad E(\mathbf{x}, \mathbf{y}) \sim |\mathbf{x} - \mathbf{y}|^{-\lambda} \quad (13)$$

with small positive values of indices $\lambda \simeq 0.2$, $0.02 < \delta < 0.07$ in accordance with [14],[15]. The Kolmogorov's theory predicts $\lambda = \delta = 0$. Now we demonstrate that in the scaling limit $\tau \rightarrow 0$, at r_∞ (i.e., when the Feigenbaum attractor of the map T loses its stability) the infinite sum in (10) provides the anomalous scaling $\lambda > \delta > 0$.

First, we show that the pair correlation $\mathcal{C}_{2\tau}$ provides the leading contribution to the sum in (10) $\propto (\tau/t - t')^\eta$. The contributions due to the triple and quadruple correlations etc. decay much more rapidly as $\tau \rightarrow 0$. To prove it in an elegant way, we use the theorem followed which is analogous to the Wick's theorem of quantum field theory: The average of the product is obtained as a sum of all possible paired averages.

Theorem

In the scaling region $\tau \rightarrow 0$, the following expansion has place

$$\mathcal{C}_{\tau n}(t; \mathbf{q}_1, \dots, \mathbf{q}_n) = \sum_{i,j=1}^n \mathcal{C}_{2\tau}(t; \mathbf{q}_i) \prod_{j>s, s \neq i} \mathcal{C}_{2\tau}(t; \mathbf{q}_j - \mathbf{q}_s).$$

The elementary proof of the theorem is given in the Appendix. This theorem gives a key to diagram technique in dynamical systems. The diagrammatic approach to computation of arbitrary correlation functions in dynamical system theory has been developed recently in [16]. One can estimate the scaling asymptotics for correlation functions as $\mathcal{C}_{\tau n} \propto \tau^{n\eta/2}$ (n is assumed to be an even number) then the result on the scaling behavior seems obvious.

Interesting in the scaling asymptotics $\tau \rightarrow 0$, we neglect all higher order correlations in (10) except $\mathcal{C}_{2\tau}$, so that, for $r \rightarrow r_\infty$, in the

inertial range, in $d = 3$, (10) reads as follows

$$K(k, t - t') = A' k^{-11/3} \left(\frac{\tau}{t - t'} \right)^\eta + \mathcal{O}(r - r_\infty), \quad (14)$$

in which A' is some modified Kolmogorov's constant. The asymptotics (14) can be treated as a consequence of modification of the critical regime in the theory (12) as the result of rescaling:

$$\tilde{v} = v\xi, \quad \tilde{v}' = v'\xi^{-3}, \quad \tilde{g} = g\xi^{2+d}, \quad \tilde{k} = k\xi, \quad \tilde{t} = t\xi^{-2}, \quad (15)$$

in which

$$\xi \equiv (\tau/t)^\sigma, \quad \sigma = \frac{\eta}{2 + d - 2\Delta_v},$$

Δ_v is the Kolmogorov's dimension of velocity. The critical exponents of all \sim -quantities are known (they equal to the Kolmogorov's values). The scaling parameter of the doubling transformation τ does not involve the theory (12), so that it has no definite critical dimension. Therefore, the modification of scaling asymptotics $s \equiv k/\mathfrak{M} \rightarrow 0$ due to (15) goes from the rescaling with respect to the dimensional parameter $t \propto s^{-2/3}$. The upper critical dimension (above which the hydrodynamical interaction becomes infrared unimportant, and the field theory is asymptotically free) for the rescaled model is changed to $4 - 4/3(2 + d)\sigma$ and is not equal to the logarithmic dimension $d_l = 4$. In dimension $d = 4 - 2\varepsilon$, the hydrodynamical interaction is important for $2/3(2 + d)\sigma < \varepsilon$ (this condition is satisfied for the actual value $\varepsilon = 2$). Thus, the quantities

$$\Delta'_k = 1 - 2/3\sigma, \quad \Delta'_v = \Delta_v - 2/3\sigma, \quad \Delta'_t = \Delta_t + 4/3\sigma, \quad (16)$$

generated by (15) are the entire critical dimensions of momentum, velocity, and time in turbulence launched by the dynamical system.

The simple power counting in (14) gives the asymptotics

$$K \sim k^{-(11/3 + \delta_\eta)}, \quad \delta_\eta = 2\eta/3\sigma \simeq 0.2014722 \dots$$

The renormalization of the local energy dissipation operator E with mixing has been given in [5] in details. It was shown that the marginal contribution into asymptotics is risen due to the linear combinations of composite operators including $\langle \mathbf{v} \Delta \mathbf{v} \rangle(\mathbf{x})$, which has the critical exponent $\Delta_E = 4 - 3\gamma_t = 0$, where $\gamma_t = 2 + \Delta_t = 4/3$ is the anomalous dimension of time in the Kolmogorov's theory. In

the rescaled theory, the anomalous dimension of time is changed to $\gamma'_t = 2 + \Delta'_t \simeq 0.7107451 \dots$, so that the resulting critical dimension $\Delta'_E \simeq 1.8677646 \dots > 0$ is no more marginal. This corresponds to the power law behavior

$$E(\mathbf{x}, \mathbf{y}) \sim |\mathbf{x} - \mathbf{y}|^{-\lambda}, \quad \lambda \simeq 1.1322354 \dots$$

In conclusion, we have studied the model of turbulence driven by the deterministic chaotic dynamics generated by a unimodular iterated map. The critical scaling is established when the Feigenbaum attractor loses stability. When the new time scale comes into play, the Kolmogorov's critical regime is modified and the anomalous exponents become positive $\lambda > \delta > 0$. However, their absolute values prescribed by the Feigenbaum constant η are larger than those in [14],[15].

Appendix

A proof sketch of the Theorem.

Let us consider the product of partial sums

$$\begin{aligned} & \mathfrak{P}(\mathbf{q}, \mathbf{q}_1 \dots \mathbf{q}_n) \\ & \equiv \frac{1}{n} \sum_{\mathbf{p}} z_{\mathbf{p}} z_{\mathbf{p}+\mathbf{q}} \frac{1}{n} \sum_{\mathbf{p}_1} z_{\mathbf{p}_1+\mathbf{q}_1} z_{\mathbf{p}_1+\mathbf{q}_2} \dots \frac{1}{n} \sum_{\mathbf{p}_n} z_{\mathbf{p}_n+\mathbf{q}_{n-1}} z_{\mathbf{p}_n+\mathbf{q}_n}. \end{aligned}$$

Rename the counting variable in each sum, for example

$$\frac{1}{n} \sum_{\mathbf{p}_1} z_{\mathbf{p}_1+\mathbf{q}_1} z_{\mathbf{p}_1+\mathbf{q}_2} = \frac{n + \mathbf{q}_1}{n} \frac{1}{n + \mathbf{q}_1} \sum_{\mathbf{r}=1+\mathbf{q}_1}^{n+\mathbf{q}_1} z_{\mathbf{r}} z_{\mathbf{r}+(\mathbf{q}_2-\mathbf{q}_1)}.$$

As $n \rightarrow \infty$, this partial sum tends to $\mathcal{C}_2(\mathbf{q}_2 - \mathbf{q}_1)$. It remains to note that

$$\mathcal{C}_n(\mathbf{q}_1, \dots, \mathbf{q}_n) = \sum_{\mathbf{q}_1 \dots \mathbf{q}_{n-1}} \delta_{\mathbf{p}\mathbf{p}_1} \dots \delta_{\mathbf{p}\mathbf{p}_{n-1}} \mathfrak{P}(\mathbf{q}, \mathbf{q}_1 \dots \mathbf{q}_n).$$

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$$Sf = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2$$

over the whole interval $[0, 1]$.

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